THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL MATHEMATICS COMPETITION

1970 EXAMINATIONS

29TH MAY

Time allowed—Three hours

Books and aids may be used. Otherwise normal examination conditions apply.

This is not a *conventional* examination. Additional credit will be given for correct solutions in which exceptional mathematical insight or ability is shown. For example a candidate may, if he is able, indicate briefly any alternative method, or any generalization of the problem and solution.

A complete solution to a single question will receive more credit than a number of fragmentary answers

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JUNIOR DIVISION EXAMINATION

Candidates may attempt *all* questions

1. Solve for x, y and z the simultaneous system of equations

$$x(x + y) + z(x - y) = a,$$
$$y(y + z) + x(y - z) = b,$$
$$z(z + x) + y(z - x) = c,$$

where a, b, and c are given positive integers.

What can you say if both positive and negative values are allowed for a, b and c?

2. (i) Which of the following four statements are true, and which are false? If a statement is false, give an example showing this. If the statement is true prove it.

   (a) If a polygon inscribed in a circle has equal sides then it has equal angles.

   (b) If a polygon inscribed in a circle has equal angles then it has equal sides.

   (c) If a polygon has an inscribed circle (all its sides are tangent to the circle) and its sides are equal then its angles are also equal.

   (d) If a polygon has an inscribed circle and its angles are equal then also its sides are equal.

   (ii) Let n be the number of sides of the polygon. For what values of n are all four statements true?

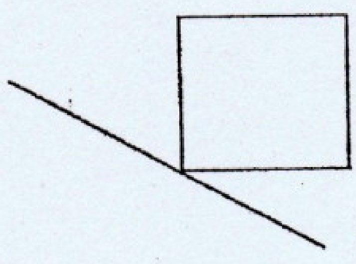
3. Prove that amongst any six consecutive positive integers there is always at least one which has no factor (apart from 1) in common with any of the others.

4. Show that the roots of the equation $ax^2 + bx + c = 0$ must be irrational if a, b and c are all odd integers. Find a cubic equation with odd integer coefficients which has rational roots.

5. You are given nine squares with sides 1, 4, 7, 8, 9, 10, 14, 15, 18 units respectively. These can be put together (without overlapping) so as to fill out a rectangle. Find the dimensions of the rectangle and show how it can be formed from the above squares.

6. A rectangle has sides m units and n units where m and n are positive integers with no common factor. The rectangle is subdivided into mn unit squares and a diagonal of the rectangle drawn. Find a formula for the number of squares traversed by the diagonal.

   Can you find a formula which will apply when m and n have a common factor? A square which has only a corner lying on the diagonal as in the accompanying diagram is not to be counted.



SENIOR DIVISION EXAMINATION

Candidates may attempt *all* questions

1. (i) Prove that amongst any six consecutive positive integers there is always at least one which is relatively prime to each of the others.

   (ii) Prove that amongst any eleven consecutive integers there is always one which is relatively prime to each of the others.

2. (i) If $P(x)$ is a polynomial of degree n whose value is an integer for $x = 0$, $x = 1$, $x = 2, \ldots$, $x = n$, then $P(x)$ is an integer when x is any integer. Prove this.

   (ii) If (i) is altered by replacing $0,1,2, \ldots ,n$ by any other $n + 1$ consecutive integers, the statement remains true. Prove this.

   (iii) $F(x)$ is a polynomial of degree n whose value is an integer for $x = 0$, $x = 1^2$, $x = 2^2, \ldots$, $x = n^2$. Prove that $F(x)$ is an integer when x is any perfect square.

3. Suppose you are provided with a straight edge with two marks on it, one unit apart. The only operations allowable with this instrument are (a) ruling straight lines and (b) marking a point on a given straight line at a unit distance from a given point on the line. (You are *not* given a compass). Using this instrument show how to perform the following constructions:

   (i) Bisect a given angle.

   (ii) Construct a square whose diagonals meet at a given point.

   (iii) Construct a square with one vertex at a given point.

4. A "graph" is a collection of a finite number of points (called "vertices") some pairs of which are joined by curves (called "edges"). There is at most one edge joining any two vertices. The only vertices on an edge are its two end points.

   A graph contains a circuit of length r if there are r vertices $P_1, P_2, \ldots, P_r$ such that each of the pairs $P_1P_2, P_2P_3, \ldots, P_{r-1}P_r$; P_rP_1 is joined by an edge. Prove that if in a graph each vertex is an end point of at least k edges ($k \geq 2$) then the graph contains a circuit of length at least $k + 1$.

5. Given a finite set S of positive integers, we call the *order* of any element x of S the length of the longest sequence of distinct elements of S starting with x, e.g. $x, y, \ldots, z$ such that each member of the sequence is a multiple of the preceding one. For example if

   $$S = \{2, 4, 5, 7, 12, 15, 16\}$$

   then the order of 2 is 3 since 2, 4, 12 or 2, 4, 16 have 3 members and there are no longer sequences of this kind starting with 2. Similarly the order of 5 is 2, the order of 12 is 1.

   Assuming that S has 101 elements and no element of S has an order exceeding 10, prove that there is a subset T of S containing at least 11 elements such that no element of T is divisible by any other member of T.

   We welcome any generalizations of this result.

V. C. N. Blight, Government Printer, New South Wales—1970