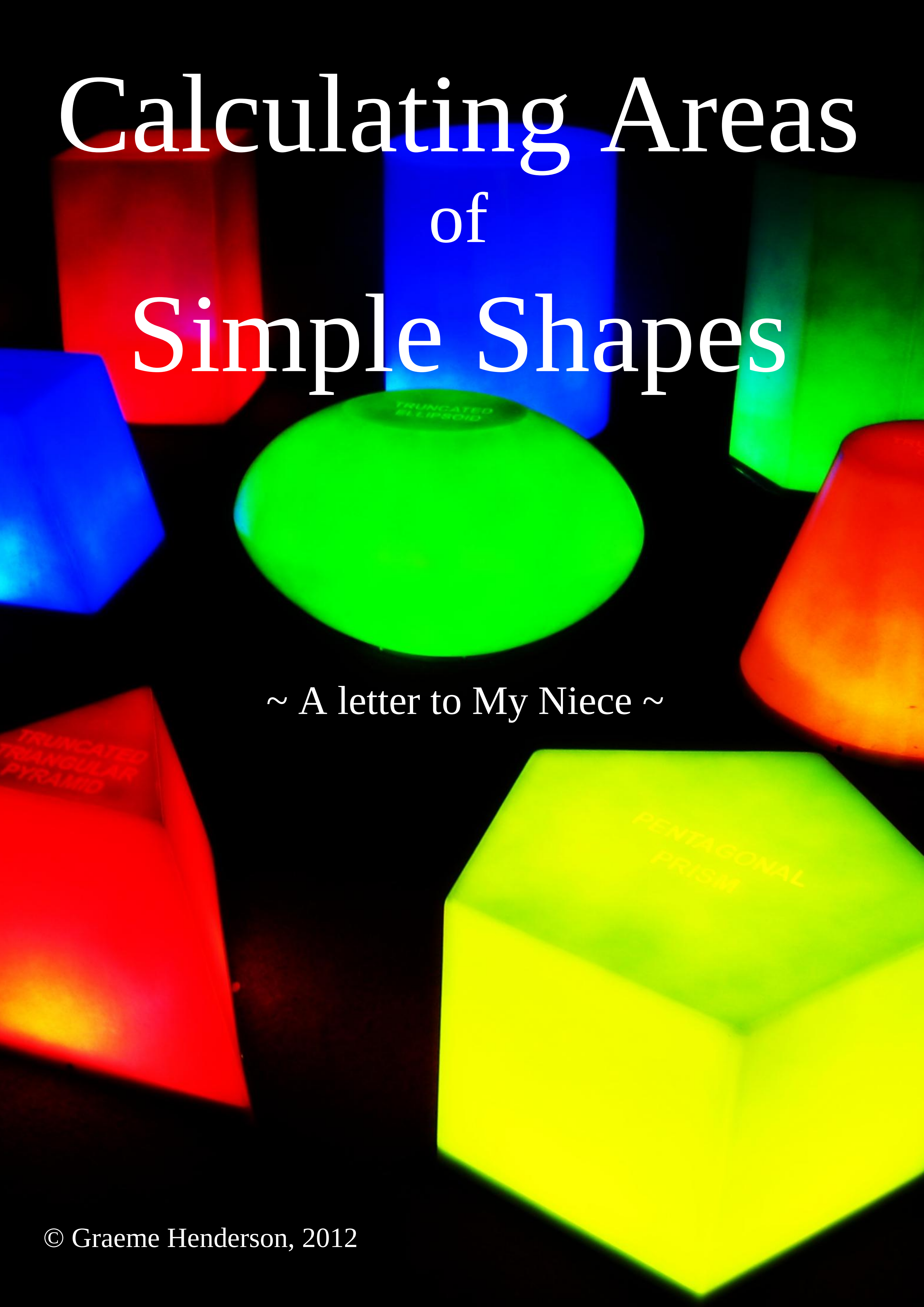
Calculating Areas of Simple Shapes

~ A letter to My Niece ~



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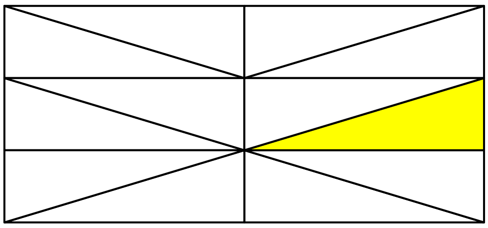
Dear Aletheia (not her real name),

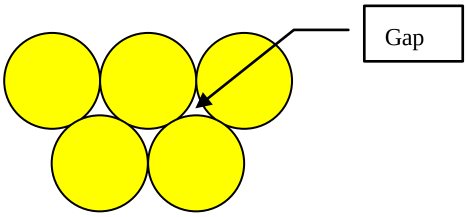
I thought I would write you some notes to help with your understanding of how to find areas of shapes.

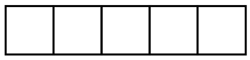
The first thing to realise is that "area" is a measure of the space inside a shape. If one shape can be repeated to make up another shape, we are able to compare their sizes.

For example, this large rectangle is twelve times the size of the shaded triangle (if all the small triangles are the same size).



Mathematicians, wanting to measure and compare the size of shapes, needed a shape and size that they could compare everything (all other shapes) to. This shape had to be "stackable" with no gaps, so circles could not be used.

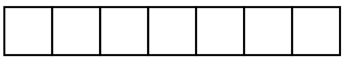


The simplest and most useful shape was the square. Accordingly,  would be said to have an area of 5 squares.

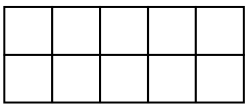
If each square was 1cm along each side it was called "one centimeter square" or "one square centimeter". The latter terminology is preferred and is written cm^2. Squares can be based on any length unit. I have even seen archaeological books referring to $cubits^2$! Older books may mention $feet^2$, $yards^2$ and $miles^2$. We use the International Metric System (SI — Système International d'Unités), so we measure areas in mm^2, cm^2, m^2, km^2, etc.

Now, if areas are to be broken into squares or, at least, measured in this way — which shapes can most easily be divided into squares, and how may this be done?

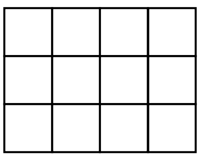
Answer: rectangles! All areas that we calculate are in some way related to rectangles (even circles!). How do we find the area of a rectangle? If it has a whole number of units on each side, the squares are easy to draw, e.g.



7 $units^2$ (7 squares)



10 $units^2$ (10 squares)



12 $units^2$ (12 squares)

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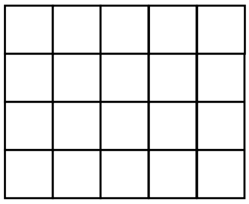
What is the short cut? When squares are neatly laid out in rows like this we simply count up the number of rows and the number of squares in each row.



We could call this a rectangle of 5 rows of four or 4 rows of five, depending on your viewpoint.

When explaining what to do, we would tell people to multiply the number of rows by the number of squares in each row!

Algebraically, we would say

	Area	equals	rows	times	number
	A	=	r	×	n
Or	Area	equals	number of rows	times	number of squares in each row
	A	=	n	×	s

In fact, whatever we call these two quantities is fine. Some textbooks have

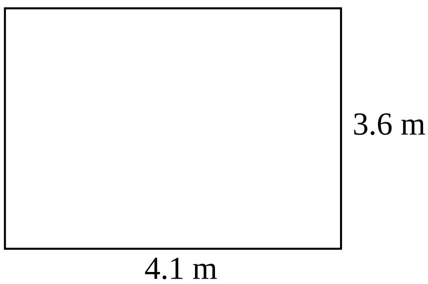
	A	=	breadth	×	height	
Others	A	=	length	×	breadth	
Others	A	=	length	×	width	etc.

My own preference is to avoid calling either figure the "height" as we use that later when finding volumes!

So,	A	=	length	×	breadth
i.e.	A	=	l	×	b
i.e.		A = lb	is my preferred formula!		

It turns out that this always works even when the length and breadth are not whole numbers. Both measurements must be in the same units.

May I recommend the following "setting out"?



$$\begin{aligned} A &= lb \\ &= 4.1\text{ m} \times 3.6\text{ m} \\ &= 14.76\text{ m}^2 \end{aligned}$$

Always start with "A =" to show what you are finding (AREA).

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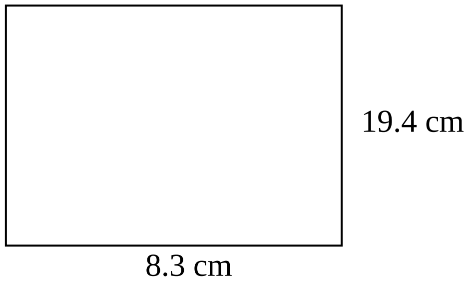
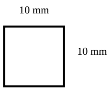
Note that the "=" signs are lined up vertically. Always remember to write the units at the end of your answer. It is good to practice this skill with quite a few rectangles so that you can find the areas quickly and without having to stop and think about anything except the numeric calculation.

Before we go further, I must add one more observation. In science, and in real life, you cannot be expected to have an answer noticeably more accurate than the information you were given. For example, to find the area of the following rectangle, we solve as follows:



$$\begin{aligned} A &= lb \\ &= 19.4\text{ cm} \times 8.3\text{ cm} \\ &= 161.02\text{ cm}^2 \end{aligned}$$

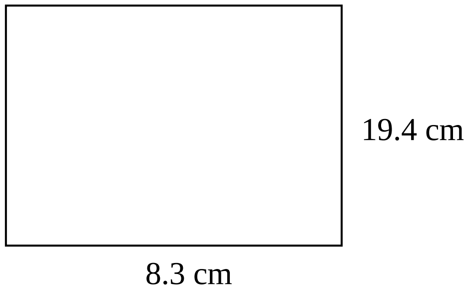
Now, each measurement has been given to one decimal place, to the <u>nearest</u> mm. This means that they <u>could</u> be up to ½ mm in error (either over or under the real measurement).

There are 100m^2 in 1cm^2 

, so the answer, 161.02 cm^2 has been given to the nearest square mm. This kind of accuracy is <u>impossible</u> if the two original measurements could be "out" by ½ mm each.

I leave it to you to see what answers you might get if the rectangle was 19.45 cm × 8.35 cm (1/2 mm bigger) or 19.35 mm × 8.25 mm (1/2 mm smaller).

The way scientists, engineers, surveyors, builders et al. deal with this problem in real life is by thinking about <u>significant figures</u>. This is a measure of how accurate a measurement is according to how many numbers are used in writing it.



19.4 cm has 3 significant figures.

8.3 cm has 2 significant figures.

The rule that is used is this — your answer should be no more accurate than the least accurate bit of information that you have been given.

In other words, the number of significant figures in your answer should be equal to the least number of significant figures in your information.

Because this means "rounding off" our answer, we no longer use the "=" sign. Instead, we use the "is approximately equal to" sign, ≈ (or I prefer $\doteqdot$ because it is also acceptable and is much easier to draw), although you will not be marked wrong in school if you do not use this sign.

Because 8.3 cm has only two significant figures we reduce our answer to two significant figures. I like to draw a vertical line where I am going to "round off" my answer. So, here is the complete working for this and one other problem.

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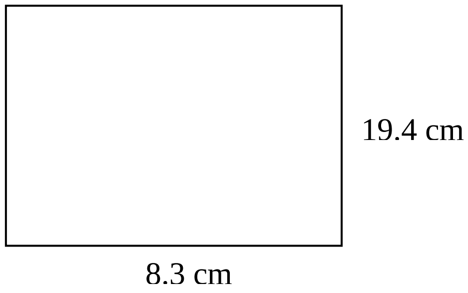
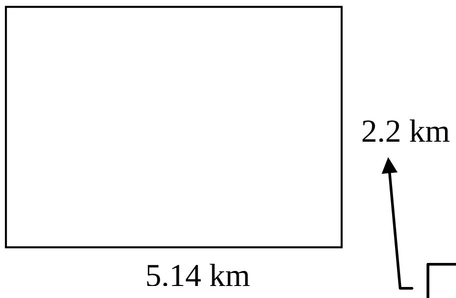


$$\begin{aligned} A &= lb \\ &= 19.4 \text{ cm} \times 8.3 \text{ cm} \\ &= 161.02 \text{ cm}^2 \\ &\approx 160 \text{ cm}^2 \end{aligned}$$



$$\begin{aligned} A &= lb \\ &= 5.14 \text{ km} \times 2.2 \text{ km} \\ &= 11.308 \text{ km}^2 \\ &\approx 11 \text{ km}^2 \end{aligned}$$

Two significant figures

I hope all this makes sense.

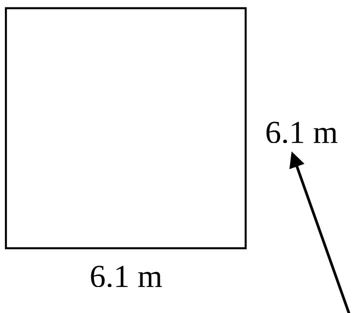
Now, how do we find areas of shapes that are "special" rectangles or not even rectangles at all?

SQUARES

Because their length and breadth are the same (and often referred to as "sides"), a lot of school text books use the formula

$$\begin{aligned} A &= \text{side} \times \text{side} \\ &= \text{side}^2 \\ &= s^2 \end{aligned}$$

I still prefer to use A = lb and write the numbers accordingly, e.g.



$$\begin{aligned} A &= lb \\ &= 6.1 \text{ m} \times 6.1 \text{ m} \\ &= (6.1 \text{ m})^2 \\ &= 37.21 \text{ m}^2 \\ &\approx 37 \text{ m}^2 \end{aligned}$$

Two significant figures

PARALLELOGRAMS

First, a note about the spelling ☺. Probably every word in English that starts with "para" comes from Greek. There is no word in English that starts with "parra" (except for a word which probably comes from Parramatta). Also, the symbol for parallel lines is $||$ and this looks like a double–ell. So I use it in para$||$el.

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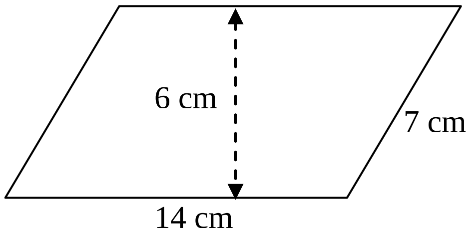
The parallelogram is made from two pairs of parallel sides. The measurements we need to find an area will always be at right angles to each other (like the sides of each square).

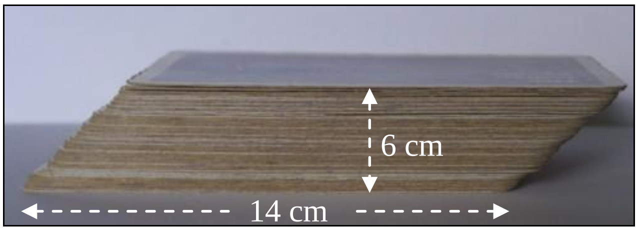
A trick that teachers use is to give the slant height as well, e.g.



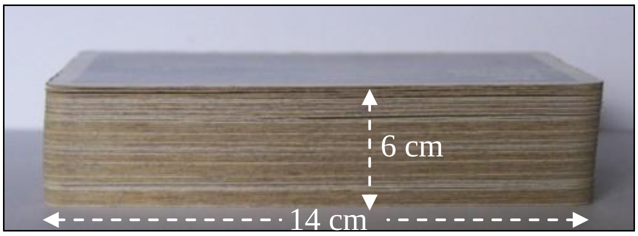
… but the 7 cm has nothing to do with finding areas. It can be used for finding other things, like perimeters but, to find the area, we need the 14cm and 6cm because they are at right angles to each other!

Now, how do we use them to find the area of the parallelogram? There are two ways of thinking about this and I will share them with you. First, imagine that the parallelogram is what you see when looking at the side/edge of a pack of cards that has been pushed over a bit.

You can think of the parallelogram as being sliced into lots and lots of very thin horizontal cards.



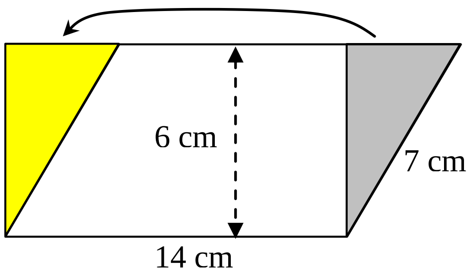
If the cards are able to slide against each other, I hope you can see that, by tapping the sides carefully, you could straighten them as shown here:



Each card would still be 14cm long, there would still be the same number of cards, the edge of each card will still be the same size, and the height of the stack/deck would still be 6cm.

The area of the parallelogram is exactly the same as the area of the rectangle with the same length and breadth! This is an amazing but very useful result.

The second way of looking at the problem is this:



You could cut off the triangle at one end (at right angles to the base) and take it to the other end where it will fit exactly (remember the parallel lines). Again, the area of the parallelogram is the same as the area of the rectangle.

Some text books use the formula A = base × height

= bh

for parallelograms! They confuse students by using three different formulae.

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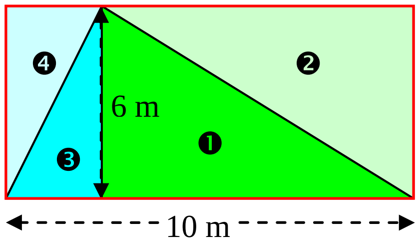
SUMMARY

SQUARE	$A = s^2$		
RECTANGLE	$A = lb$	$\Big\}$	All three are "rectangles," i.e. $A = lb$
PARALLELOGRAM	$A = bh$		

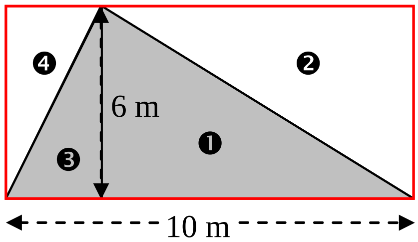
Now for the next three shapes!

TRIANGLES

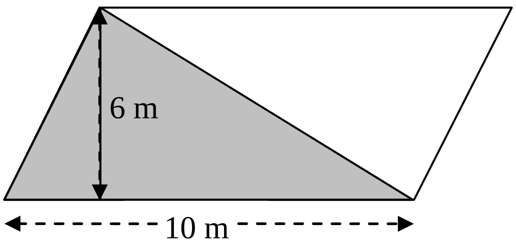
If you build a rectangle (in red) around the triangle you can see that area (1) equals area (2) and area (3) equals area (4).



This means that the triangle (shaded below) is exactly half of the rectangle!



Another way of thinking of this is that the triangle is also exactly half of a parallelogram (and that the parallelogram has the same area as the rectangle).



Triangle = half parallelogram

= half rectangle

Text books often like using $A_\Delta = \dfrac{base \times height}{2}$

$= \dfrac{bh}{2}$

but I prefer $A_\Delta = \dfrac{lb}{2}$

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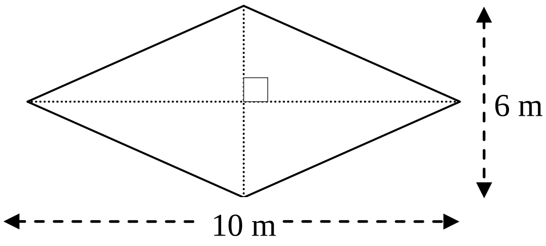
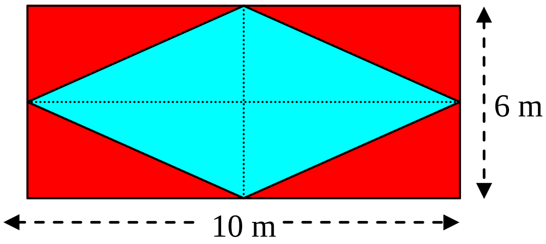
RHOMBUSES (four equal sides)

These are usually measured from tip to tip.



Note that the two lengths cross at right angles as they always must to find areas.

Let's draw a rectangle around this rhombus with the same length and breadth.



I hope you can see that the four inside (blue) triangles exactly match the four outside (red) triangles.

The blue rhombus has an area exactly half that of the rectangle.

A lot of text books use the formula:

$$A = \frac{xy}{2} = \frac{1}{2}xy$$

or they might use

$$A = \frac{ab}{2} = \frac{1}{2}ab$$

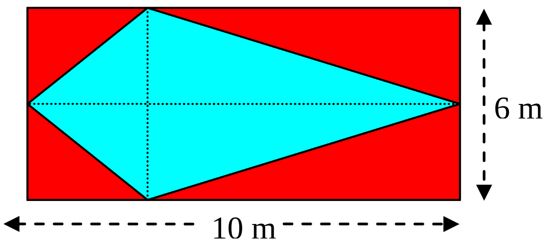
but I prefer

$$A = \frac{lb}{2}$$

since the area of a rectangle is A = lb and this is half a rectangle!

KITES

In the same way, I hope you can see that the area of the kite is half the area of the red rectangle with the same length and breadth!



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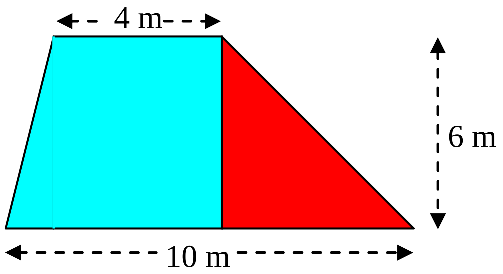
SUMMARY

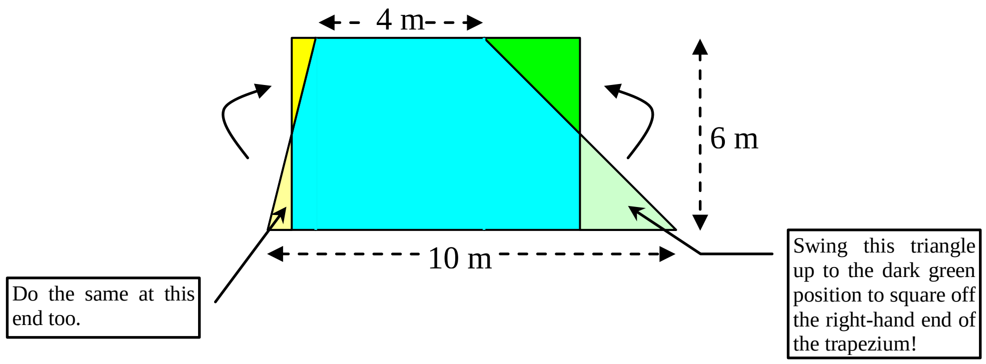
TRIANGLES	A = bh/2	All three are "half rectangles," i.e. $A = \dfrac{lb}{2}$
RHOMBUSES	A = xy/2	
KITES	A = ab/2	

FINALLY, THE TRAPEZIUM

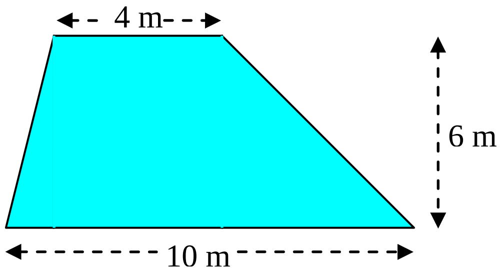
Unfortunately, we cannot cut the triangle from one end and take it to the other end (as we did with the parallelogram) because it is the wrong size/shape and would not fit!



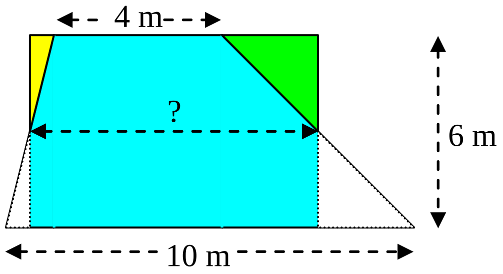
But some smart person worked out that, if you found the halfway position down one side, and cut a triangle off from there, it could be used to fill in the missing bit on the same end!



So, a trapezium like this …



has exactly the same area as the rectangle:



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You can see that the height is still 6 m. It has not changed.

But what is the length of the new rectangle? It turns out that because the line with the question mark is exactly halfway between the top and bottom lines (and because the two sloping lines are straight) that the length of the rectangle is exactly halfway between the length of the top and bottom sides of the trapezium.

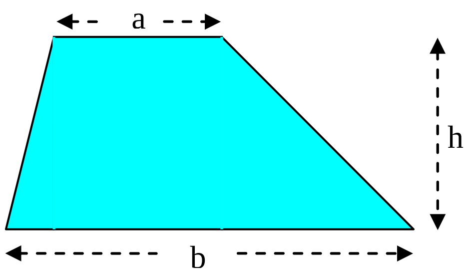
I am sure you know what number is exactly halfway between 4 m and 10 m. This means that the rectangle is 7 m long. Since we know that it is also 6 m high/broad, the area of the trapezium and rectangle is

$$\begin{aligned} A &= lb \\ &= 7\text{ m} \times 6\text{ m} \\ &= 42\text{ m}^2 \end{aligned}$$

But how would you find the number halfway between difficult numbers like 21.6 and 57.04? It turns out that the number halfway between two numbers is their average.

So, the area of the trapezium is the same as the area of the rectangle with the same height and the average length.

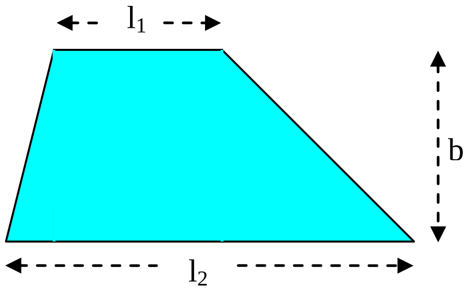
This can be written in a variety of ways:



$$\begin{aligned} A &= \frac{(a+b)}{2}h \\ &= h\frac{(a+b)}{2} \\ &= \frac{h}{2}(a+b) \\ &= \frac{1}{2}h(a+b) \end{aligned}$$

The last form is the most common one in text books.

I like to use length #1 (l_1) and length #2 (l_2), and breadth (b).



$$\begin{aligned} A &= \frac{(l_1+l_2)}{2}b \\ &= \bar{l}b \end{aligned}$$

Where $\bar{l}$ means average length,” i.e. $\bar{l} = \dfrac{l_1+l_2}{2}$

So, finally, after a lot of pages, we can write our summary over the page.

Please learn this summary, Aletheia. I hope it simplifies things and helps.

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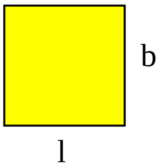
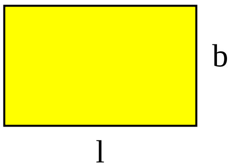
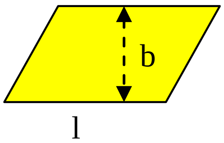
AREA SUMMARY



SQUARE

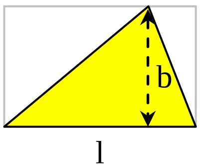


RECTANGLE

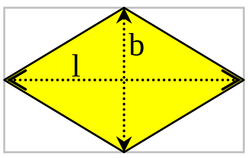


PARALLELOGRAM

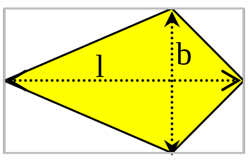
All three are "rectangles," i.e. $A = lb$



TRIANGLE

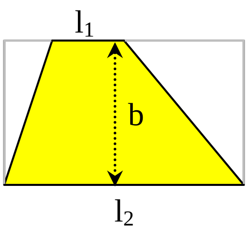


RHOMBUS



KITE

All three are "half rectangles," i.e. $A = \frac{lb}{2}$

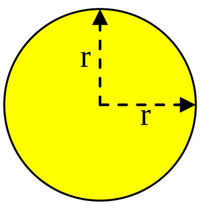


TRAPEZIUM
(TRAPEZOID IN USA)

This is an "average rectangle," i.e. $A = \bar{l}b$

On another occasion I will share with you how circles can be rearranged to look like rectangles too!

So that this can be an even more useful summary page, I will include the formula for the area of a circle as well.



CIRCLE

Even this can form a rectangle … $A = \pi r^2$

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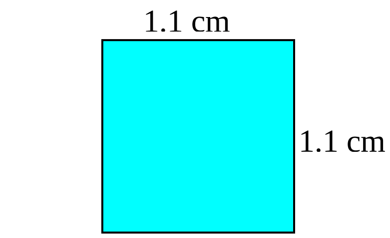
Examples of Use (I found these Qs in a school textbook):

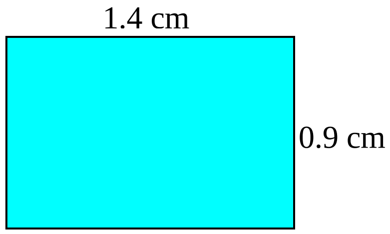


$A = lb$

$= 1.1\text{ cm} \times 1.1\text{ cm}$

$= 1.21\text{ cm}^2$

$\approx 1.2\text{ cm}^2$

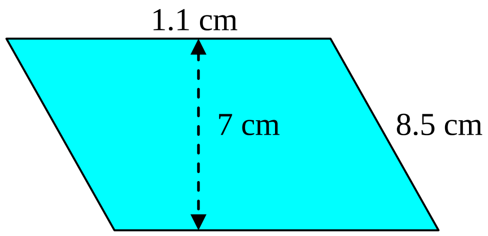


$A = lb$

$= 1.4\text{ cm} \times 0.9\text{ cm}$

$= 1.26\text{ cm}^2$

$\approx 1.3\text{ cm}^2$

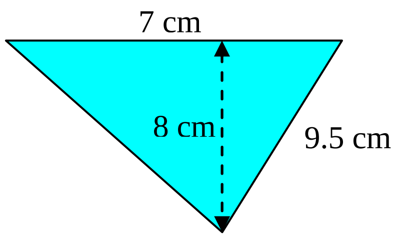


$A = lb$

$= 1.1\text{ cm} \times 7\text{ cm}$

$= 7.7\text{ cm}^2$

(8.5 cm is not at right angles to the length)

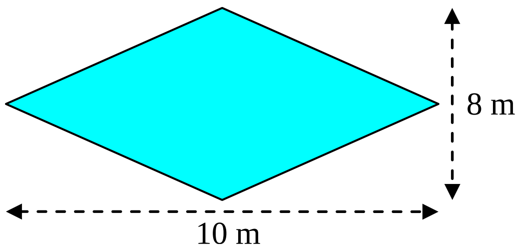


$$A = \frac{lb}{2}$$

$$= \frac{7cm \times 8cm}{2}$$

$= 28\text{ cm}^2$

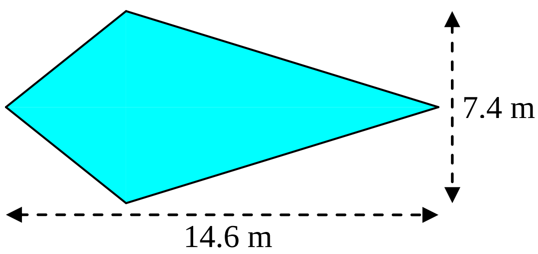
(9.5 cm is not at right angles to the length)



$$A = \frac{lb}{2}$$

$$= \frac{10m \times 8m}{2}$$

$= 40\text{ m}^2$



$$A = \frac{lb}{2}$$

$$= \frac{14.6m \times 7.4m}{2}$$

$= 54.02\text{ m}^2$

$\approx 54\text{ m}^2$

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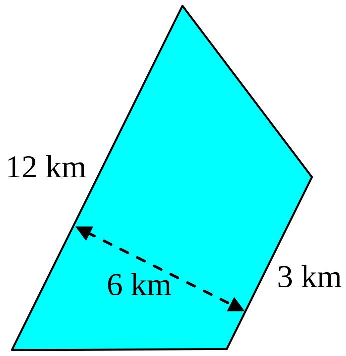


$$A = \left(\frac{l_1 + l_2}{2}\right)b$$

$$= \left(\frac{12km + 3km}{2}\right)6km$$

$$= 45 \text{ km}^2$$

Easy, isn't it? I hope so.

Now I will share just a little about composite areas before I say goodbye.

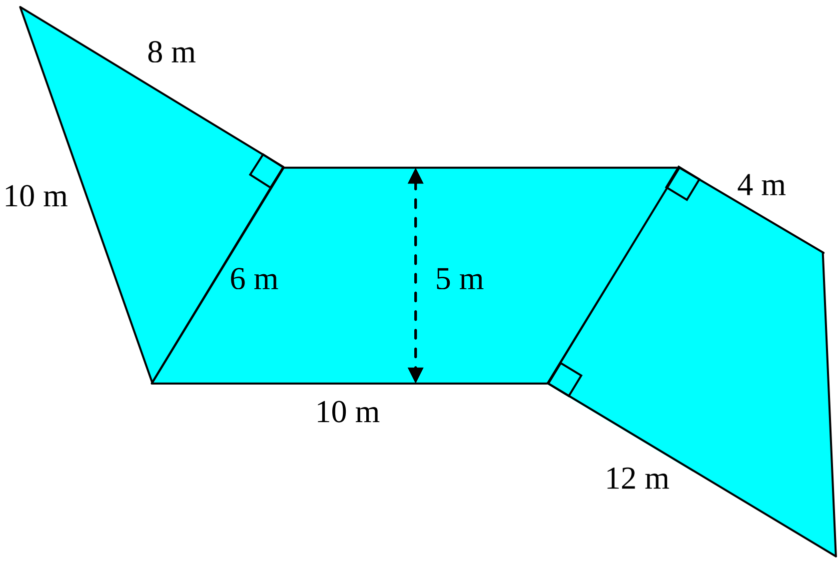
To find the area of composite shapes, it can really help if you learn to use subscripts.

"Sub" means "under" and "scriptum" means "writing" — so subscripts are things we write under the line that do not get involved in the calculations but explain what is going on.

I have already used subscripts. For example, l_1 means length #1 and A_Δ means the area inside a triangle.

In the same way we can write "total area" as A_{TOT} or "Area #1" as A_1 or "Area of square" as $A_\square$.

Let's imagine that we have to find this area:



It looks awful, but it is just a triangle, a parallelogram and a trapezium.

Note how I set my work out and use the formulae for each shape. I hope it makes sense to you. I have used words as subscripts when I could not find appropriate shapes, but the shapes are easier to draw.

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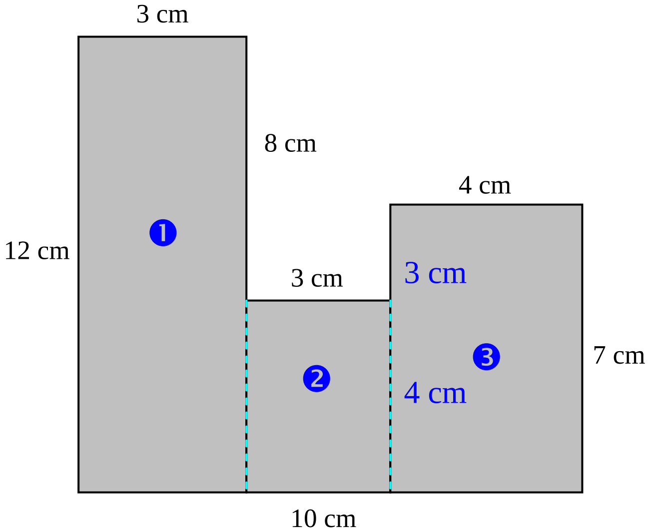
$$A_{TOT} = A_{\Delta} + A_{Parallelogram} + A_{Trapezium}$$
$$= \frac{lb}{2} + lb + \left(\frac{l_1 + l_2}{2}\right)b$$
$$= \left(\frac{8\times 6}{2}\right) + (10\times 5) + \left(\frac{4+12}{2}\right)6$$
$$= 24 + 50 + 48$$
$$= 122 \text{ m}^2$$

$\approx 120 \text{ m}^2$ (as your information was only accurate to 1 or 2 significant figures[1])

Even very complicated shapes can be studied this way. If you have a few shapes the same (like rectangles) it can be a good idea to label your diagram with numbers so that you don't get confused concerning the shapes. Here is an example to show you what I mean:



Our first job is to find the missing lengths. I have shown these in blue.

The horizontal 3 + 3 + 4 at the top of each rectangle have to add to the total horizontal width of 10 cm shown at the bottom.

The vertical lengths have to add to 12cm but the 8 cm and 7 cm lengths overlap (because the last rectangle is taller than the middle one), so we calculate 8 cm + 7 cm – 12 cm = 3 cm overlap.

The second job is to divide the shape into simpler bits. I have chosen the three rectangles (labelled in blue). Here is how I now find the total area:

$$A_{TOT} = A_1 + A_2 + A_3$$
$$= lb_1 + lb_2 + lb_3$$
$$= (12\times 3) + (3\times 4) + (4\times 7)$$
$$= 36 + 12 + 28$$
$$= 76 \text{ cm}^2$$

[1] The matter of accuracy of answers is a little more complicated than I have shared with you. We can discuss this at another time.

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I had hoped to explain this face to face. It takes so many pages explaining everything in writing, but I hope it was worth your while.

Best wishes from Uncle Graeme

Written on 20th August, 2012 — the day of your great-grandmother's 102nd birthday!

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