8.—GRAPHICAL DIFFERENTIATION

Some years ago as an engineering student, I was fascinated when introduced to graphical differentiation (and integration). As I have not seen the method treated in textbooks on pure mathematics, and because I consider it helpful to fourth and fifth year pupils studying calculus I am reproducing the method here.

The construction depends on the assumption that if ordinates are drawn through two points on a curve taken reasonably close together, the tangent at the mid-ordinate is approximately parallel to the chord joining the points.

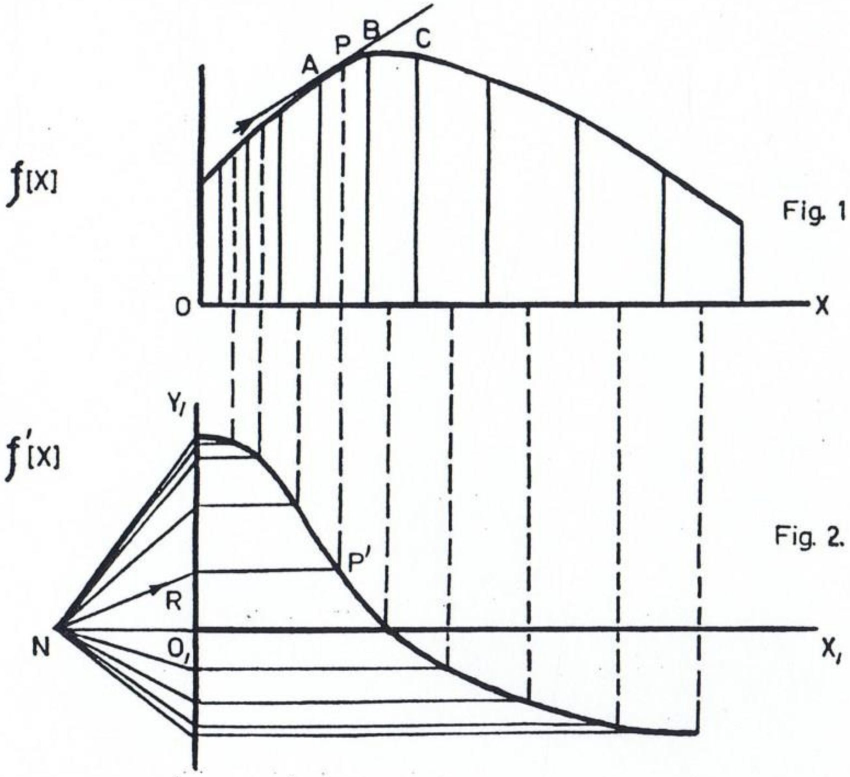

Fig. 1

Fig. 2.

Construction—

1. The diagram is divided into a number of vertical strips as in fig. 1. The interval between these strips may be varied from point to point along the curve. Where the radius of curvature is small, the interval may be shortened in order to increase the accuracy. On the other hand, where the radius of curvature is large, a larger interval may be taken without sacrificing appreciably the accuracy of the results obtained.
2. Choose a convenient pole N on X_1O_1 produced. Draw NR parallel to the chord AB to intersect the Y-axis in R. Produce the ordinate at P to meet a horizontal from R in P^1. The point of intersection P^1 lies on the derived curve. By determining a sufficient number of points the derived curve may be drawn.

It is preferable to choose the point N so that O_1N represents a whole number of units on the scale chosen and also to make the line drawn through N parallel to the tangent at the point of maximum slope intercept a convenient length on the Y-axis.

Graphical differentiation is frequently used by engineers who require the derived curves (*e.g.*, to calculate speeds and accelerations) when the algebraic equation connecting two functions is not known. Apart from this practical application, I consider the method helps pupils to grasp more fully the geometrical significance of the first and second derivatives.

Long before treating the differentiation of trigonometrical functions in the usual way, pupils can obtain the result

$$\frac{d\,(\sin x)}{dx} = \cos x$$

by differentiating graphically the sine curve when the readily recognised cosine curve is obtained.

Turning Points and Points of Inflexion

Graphical differentiation can be of great value when discussing turning points and points of inflexion. Pupils can be given a curve such as in figure 3 and can reproduce for themselves, without much difficulty, figures 4 and 5 shown. These diagrams are most valuable aids in discussing this topic.

It can be readily seen from the diagrams that:—

(i) For a *minimum value* as at the point A—

(*a*) $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2}$ is positive,

or

(*b*) as x increases $\dfrac{dy}{dx}$ behaves —, 0, +.

(ii) For a *maximum value* as at C similar results follow.

(iii) At *points of inflexion* such as B, D, E, as x increases

$$\frac{d^2y}{dx^2} \text{ behaves } +,\ 0,\ - \text{ or } -,\ 0,\ +.$$

Hence the rule for points of inflexion : d^2y/dx^2 changes sign through 0.

By comparing fig. 3 and fig. 4 it can be seen that a point of inflexion occurs in $f(x)$ whenever $f'(x)$ has a maximum or a minimum value. Hence the rule (iii) for finding points of inflexion is really the same as rule (i) (*b*) for finding a minimum (or maximum) value applied to the first derived curve.

i.e., at points of inflexion, as x increases

$$\frac{d}{dx}[f'(x)] \text{ behaves } +,\ 0,\ - \text{ or } -,\ 0,\ +.$$

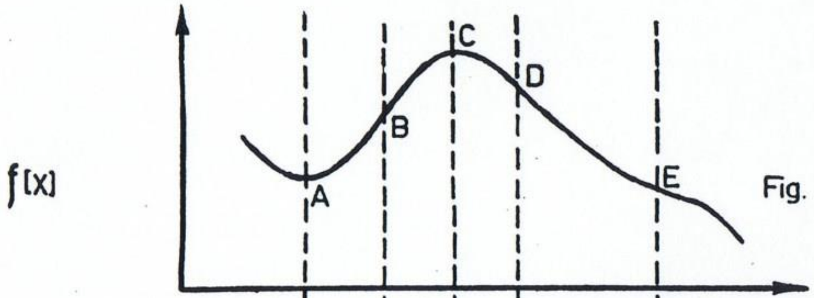

Fig. 3.

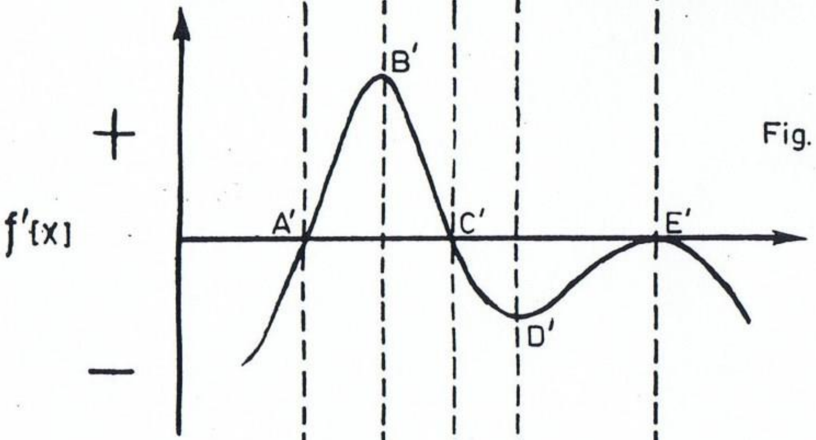

Fig. 4.

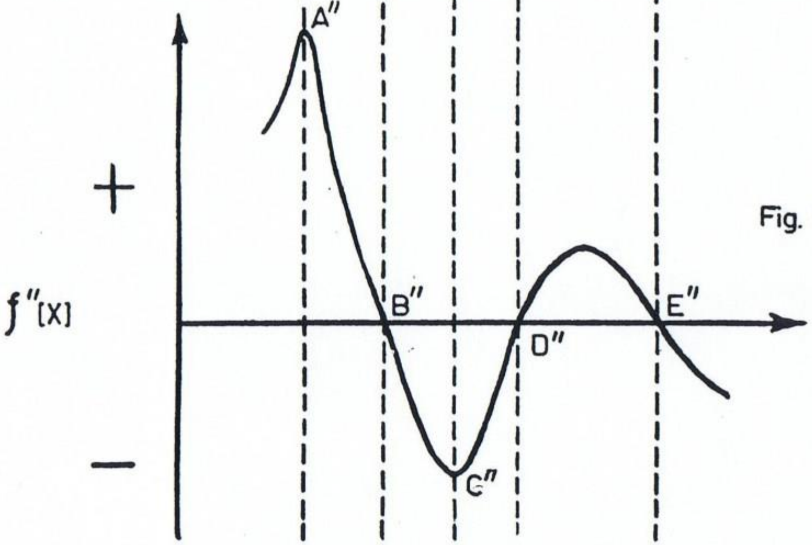

Fig. 5.

Nature of the Change of Curvature

The brighter pupil will be able to see that when d^2y/dx^2 behaves +, 0, — as x increases, the curvature changes from being convex to the axis of X to concave, and that when d^2y/dx^2 behaves —, 0, +, the curvature changes from concave to convex.

Alternatively for a change of curvature from convex to the axis of X to concave

$$\frac{d^2y}{dx^2} = 0$$

and

$$\frac{d^3y}{dx^3} \text{ is negative.}$$

For a change of curvature from concave to convex

$$\frac{d^2y}{dx^2} = 0$$

and

$$\frac{d^3y}{dx^3} \text{ is positive.}$$

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