Reflections on Playing 52 Pickup

I remember being introduced to the card game of 52 pickup.

Some boys and I had been playing with a deck of cards at a school camp and were packing up, getting ready for bed. One student approached me, asking if I would like to play "52 Pickup." I agreed, whereupon he tossed the pack high into the air, scattering cards in all directions. As he departed, he called out, over his shoulder, "Pick them up!" It was a hard lesson … but I was never caught out with that trick again.

I learned a few lessons that night. One of the less painful ones was that there are exactly 52 cards in a complete deck (excluding the jokers).

All Good Mathematics Begins (and Develops) With Good Questions

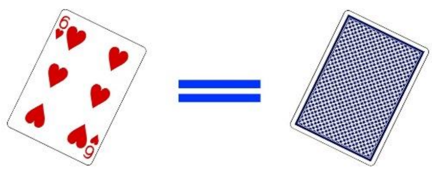
Many years later (today to be exact), I wondered in how many different ways those 52 cards could fall. This is not a trivial task. Before answering such a question, we must clearly understand what the question is asking. I am going to phrase it in more mathematical language … "How many different arrangements are possible when 52 cards are scattered over a horizontal surface?"

In this paper, we are going to answer that question. But we are going to have to clarify a few things first!

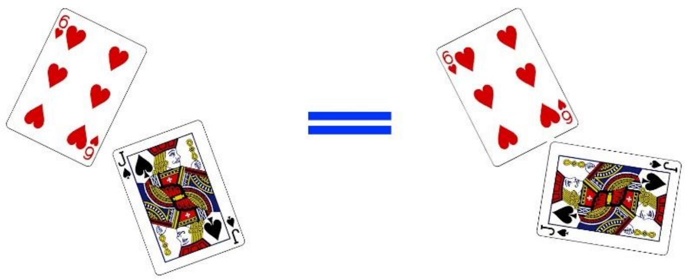
All Good Mathematics Involves Carefully Defining the Conditions (or Rules) of the 'Game'

What do we mean by an "arrangement?" What do we mean by a "location?" We must make these concepts clear in our mind before we begin to "count" such things. Since I asked the question and it is my problem, I will decide what an arrangement IS and what it ISN'T (and what a location is) … in other words, what is important and what is irrelevant to the problem:

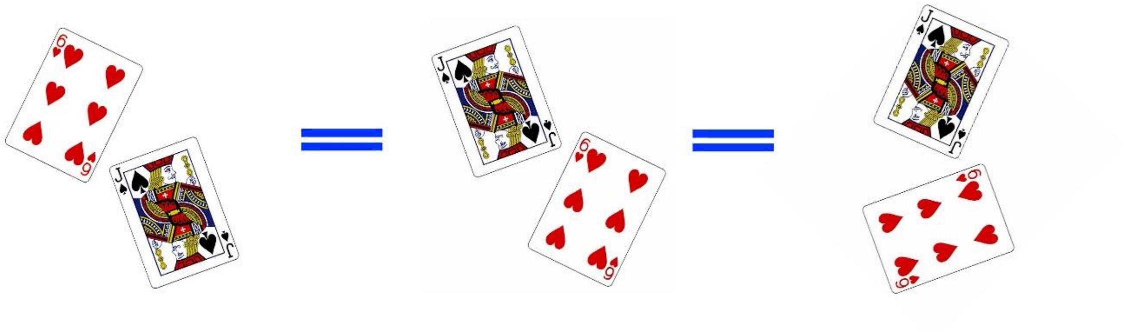
- It is not important whether a card is face up or face down, as long as I know its identity.



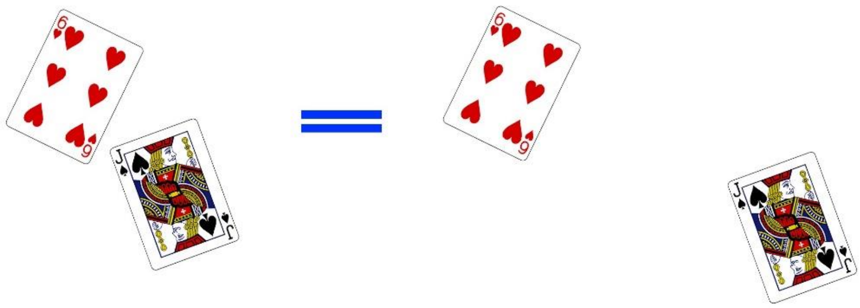
- The orientation of each card is not important. I really don't care if the long axis of one is horizontal and another is vertical relative to where I am standing. I will only be concerned with the location of the exact centre of each card, not the portions of the card around it. In this way, I can think of each card as being represented by a point, located where its centre lands. The centre of a card is where its two axes intersect/cross.



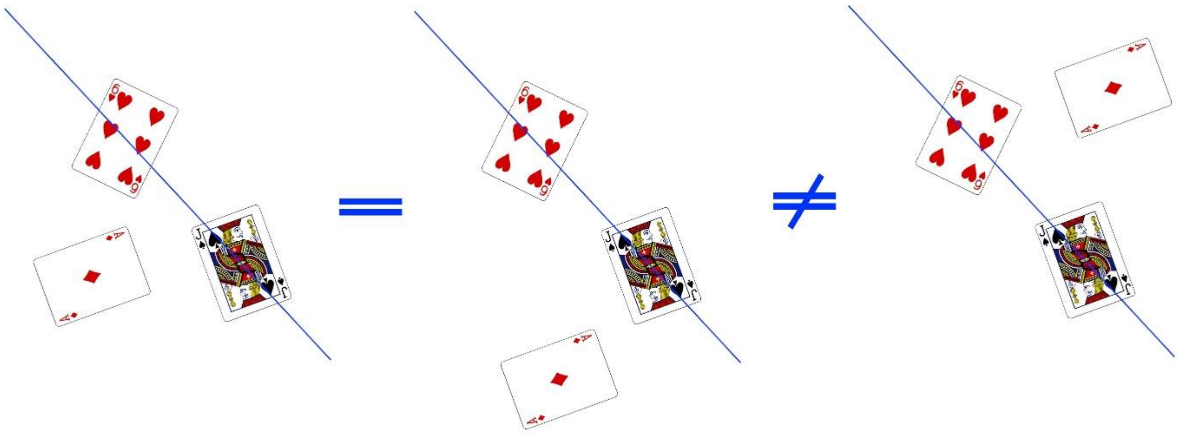
- I am concerned with the arrangement on the ground, and not with my viewpoint. For example, if I drop two cards and the red one lands to the left of the black card, I have only to walk around to the other side and the black card would then be to the left of the red one. Should I consider these as two separate arrangements (according to my point of view) or as just one arrangement (according to their arrangement on the ground)? I decide to call this one arrangement. It will simplify our calculations.



- I am not concerned with how far apart the cards are. For example, if I drop two cards and they land overlapping (with centres located, say, 2 cm apart) and I drop the same two cards again, so that they are 2 metres apart, I would call these the same arrangement.



- I am going to exclude the possibility that cards may fall so that their centres line up or so that one lands exactly on top of another. These possibilities would complicate our calculations enormously. Not only does this simplification make our calculations much easier to manage, but it could even be fairly realistic. If we were prepared to measure the location of each card to within a micrometre or a nanometre, it is highly unlikely that three cards would be collinear (line up) or that two would share exactly the same location. [Just how likely/unlikely would that be? You may like to ponder how you would calculate the answer to that question! It will involve trigonometry.]
- For similar reasons, I will also exclude the possibility that two lines formed by two pairs of cards (four cards) might be parallel. I will argue that, given a sufficiently accurate measuring system, the lines will be found to be convergent. This means that no two lines will be parallel. In other words, any pair of lines will intersect somewhere. This, also, simplifies our calculations.
- And, if a card needs to be in a particular region/location to create a new arrangement, I will not be concerned about where in that region it is. I will simply be satisfied that it is there. For example, we have worked out that there is only one arrangement when dropping two cards. They are separate. That is it! Being on the left or right is irrelevant. When I drop a third card, however (see image on next page), imagine a line drawn through the centres of the first two cards. If the new card (Ace of Diamonds) can't land in line with the previous two cards, or exactly on top of either of them, then it must land on one side or the other of that line. If we think of dropping the Six of Hearths, and then a Jack of Spades and then the Ace of Diamonds in order, we would finish with two possible arrangements. One arrangement would have the cards increasing in value (A, 6, J) in a clockwise manner. The other arrangement would have them increasing in value in an anticlockwise (counter-clockwise) manner. For the first arrangement, it is sufficient that the Ace of Diamonds has landed to the left of the line. I am not concerned with WHERE it has landed in that half-plane.

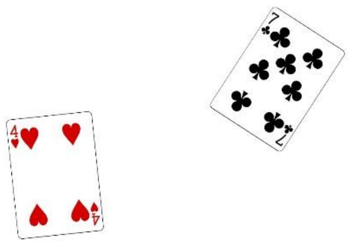


We will discover that being careful and clear with our decisions makes a big difference to how we understand and solve the problem!

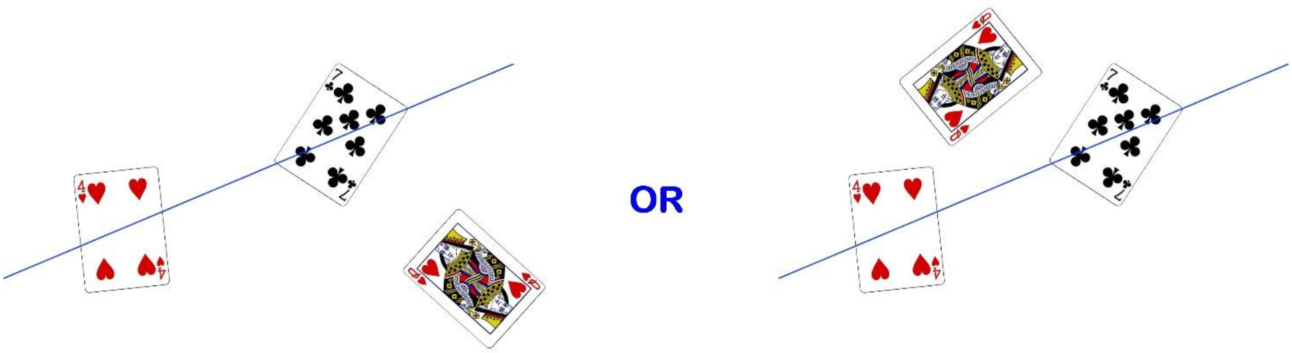
Often It Helps To Start 'Small'

Let's begin by dropping one card. I think we can agree that there is one arrangement (since we do not have to relate it to any other card).

We now drop a second card. Since we have agreed that we are not worried about distance or orientation, or how the arrangement looks from different angles, we have agreed that there is only one arrangement here, too.

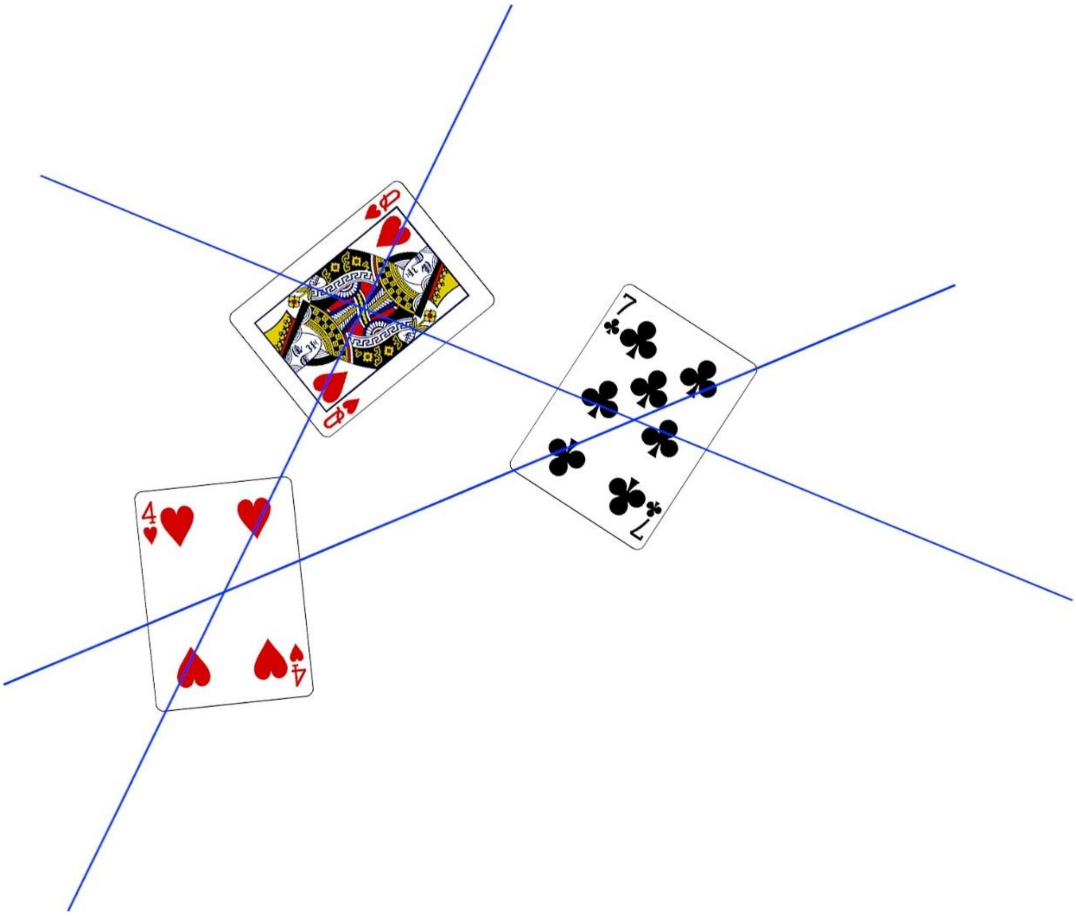


Now let's drop a third card. As shown in the image at the top of this page (and already discussed), the third card could fall into one of two regions … on one side of the line drawn between the first two cards, or on the other side. This means that we have two arrangements!



So far, things look simple.

We now drop a fourth card. Where could it land so as to create different "arrangements?" Let's start by choosing one of the previous two arrangements. I will select the one on the right (as shown in the diagram at the bottom of the previous page). The fourth card could land either side of the line joining the Queen and Seven, either side of the line joining the Seven and Four, and either side of the line joining the Queen and Four. If I draw the three lines in blue, you can see that they create seven regions. In other words, there are seven distinct regions into which the fourth card could be dropped. This analysis will apply to the anticlockwise (counter-clockwise) arrangement on the left of the previous diagram as well.



So, for four cards, we could create $2 \times 7 = 14$ different arrangements. Aren't you glad that we made some decisions that "simplified" the problem?

What have we just discovered?

What Principles Have We Learned That We Can Apply to the Next Steps?

First, it looks as though these numbers are going to grow rapidly! We have started with 1, 1, 2, and 14. This is not a principle, but it is in interesting observation.

Second, we discovered, when analysing the dropping of the fourth card, that calculating the new number of arrangements involved a product! We had to multiply the previous number of arrangements (what we had BEFORE dropping the card) by the number of different regions/positions that the new card could fall into.

Third, dropping that last card made us realise that we are really talking about REGIONS where a card may fall. This means that we will have to, somehow, calculate the number of regions each time.

Fourth, we observe that the number of regions will depend upon the number of LINES between points/cards.

Fifth, we observe that the number of lines will depend upon the number of points.

- **Can we calculate the number of regions if we know the number of points?**

If no three points are in a straight line (and we have agreed on that), then the number of regions can be found by drawing diagrams … and the formula may be deduced, especially if you know how to use Babylonian differencing. Let's draw up a table.

No. of points (n)	0	1	2	3	4	5	6	7
No. of regions (r)								

I am sure that we can agree that, if there is one point, there is one region.

It is also obvious that there is just one line joining two points (we accept Euclidean geometry here) and, therefore two regions.

If you draw three points, you will obtain three lines and seven regions, as we discussed at the top of this page.

By drawing a diagram carefully, we discover that four points will produce six lines and eighteen regions. And, by drawing a very neat diagram on A3 paper, I discovered that five points produce 41 regions, and six points produce 85 regions.

This means that we can now fill in our table and then calculate the differences (in blue). Such a table will help us create a formula for the number of regions based on the number of points!

No. of points (n)	0	1	2	3	4	5	6	7
No. of regions (r)		1	2	7	18	41	85	

1 5 11 23 44
4 6 12 21
2 6
4

So far, no pattern is emerging, but, in drawing the diagrams, I began to discern a pattern (this is why drawing diagrams can often be so helpful). I realised that, as I drew a line from the new point through existing points, it had to cross every line that already existed! Remember that we agreed that no two lines could be exactly parallel.

Admittedly, the new line would intersect a cluster of existing lines that all passed through the previously existing point/card. How many lines would be doing this? Well, if there were n points/cards on the floor, we would draw $n - 1$ lines through each point (one from each of the other points/cards).

This means we have this situation:

1. The first new line will pass through the new point and an existing point. That means that it will cross all $n(n-1)/2$ lines (the total number of lines connecting n points). But, since $n - 1$ of them passed through the existing point, we can count them as one line, because cutting them makes no new regions. In other words, we can ignore $n - 2$ lines because they will not contribute to new regions. This means that the line crosses $n(n - 1)/2 - (n - 2)$ lines.

   Now, since a line crossing a number of other lines will create one more region than the number of lines, this new line will therefore produce $n(n - 1)/2 - (n - 2) + 1$ regions.

   This simplifies to $(n^2-n)/2 - n + 2 + 1 = (n^2-n)/2 - 2n/2 + 3 = (n^2-3n)/2 + 3 = n(n - 3)/2 + 3$ regions for the first line.

2. Now we turn our attention to drawing the second line. Remember that there is now an extra line on the diagram. I.e. there are now $n(n - 1)/2 + 1$ lines. This new line will pass through

all of them but will still pass through another existing point, so we can therefore ignore n – 2 lines (as we did with the first line). This line will therefore create one more region than the first line did. I.e., it will create n(n – 3)/2 + 4 regions.

3. When we come to draw the third line, we inherits a new problem. Not only are there now an extra two lines on the diagram, two of them now cross at the NEW point! This means that one of them can be ignored because it will not contribute to creating a new region. So, we now have n(n – 1)/2 +2 lines since we have added two new ones already.

   Because this third line will pass through an existing point, we can ignore n – 2 lines as we have done in the previous two instances. But, because the new line will also pass through the new point, and there are now two lines passing through it, we have to ignore one of those as well (in order to calculate the number of regions that will be created)!

   This means that we gained a line (the second one that we just drew) but we also had to ignore a line (the same line, since it also passes through the new point … after, all that is the point through which all the new lines are being drawn). This means that the third line will create the same number of new regions as the second.

4. The fourth line now has an extra line to pass through but, since that new line also passes through the new point, it will not help create a new region … so this fourth line will create the same number of new regions as the second and third lines did!

5. This reasoning applies to all new lines drawn. They will have one more line to intersect than the previous line but, because the previous line also passed through all the other new lines at the new point, it will not help create new regions.

This means that the first new line will create n(n – 3)/2 + 3 regions, but all subsequent lines will create n(n – 3)/2 + 4 regions. Since there will be n new lines (one through each of the existing points), the total number of new regions will be n[n(n – 3)/2 + 4] – 1.

Let's Now Test Our Calculations/Analysis

We can test this formula! Substitute n = 2, 3, 4, and 5 into the 'Extra Region Formula' that we deduced …

$$r = n\left(\frac{n(n-3)}{2} + 4\right) - 1$$

… and you will obtain the results 5, 11, 23, and 44.

As you can see, this conforms to the first row of differences in the table that we created from our drawings. And these differences are the new number of regions being added each time. Therefore, we have theoretical confirmation for our table … as well as a working formula!

First of all we need to address a potential problem. Why does the formula not work with n = 1? You may have noticed that, if we substitute n = 1, we get two (2) regions, and not one as we would expect. This difference occurs because the reasoning that we used to create our formula applies to when already had points connected by lines and therefore added at least a first and second line.

When there is only one existing point, however, there are no lines crossing through it and we cannot deduce that we can ignore "one of them" and therefore subtract one line from our calculation!

In other words, this logic that we used will work for the remainder of the table but it will fail to predict the first two simple entries. But, we already know the number of regions for those two situations anyway.

Knowing the differences for the remainder of the table allows me to construct an equation for the number of regions for n > 2. Because the differences form a cubic equation { n[n(n – 3)/2 + 4] – 1 }, I know the number of regions will be a quartic equation. Let's find it by reconstructing the initial values and using the values on the leading diagonal line. Remember, the formula that we get will not work for n = 0 and n = 1, but it WILL work for every value of n > 1.

No. of points (n)	0	1	2	3	4	5	6	7
No. of regions (r)	1	0	2	7	18	41	85	

-1 **2** **5** **11** **23** **44** **77**

3 **3** **6** **12** **21** **33**

0 **3** **6** **9** **12**

3 **3** **3** **3**

The equation for the number of regions is, therefore,

$$r = 1.\,{}^{n}C_0 - 1.\,{}^{n}C_1 + 3.\,{}^{n}C_2 + 0.\,{}^{n}C_3 + 3.\,{}^{n}C_4$$

I will spare you the details, but this simplifies to $(n - 1)(n^3 - 5n^2 + 18n - 8)$.

We have found our quartic equation for the number of new regions for each arrangement!

$$Number\ of\ Regions = (n-1)(n^3 - 5n^2 + 18n - 8)$$

We Now Calculate the Number of Arrangements Using a Spreadsheet

You will remember that the first two lines do not conform to the pattern that we have worked out for n > 1, so we will simply write their figures in. But, from the third line we can design the spreadsheet as you see below. Remember that the new number of arrangements is always the number of previous arrangements multiplied by the number of possible regions for the new card to drop into.

Number of Cards /Points	Number of Arrangements	Number of Regions
(A1) = 1	1	1
(A2) = A1+1	(B2) = 1	(C2) = 2
(A3) = A2+1	(B3) = B2*C2	(C3) = (A3-1)*[A3^3-5*A3^2+18*A3-8]/8
(A4) = A3+1	(B4) = B3*C3	(C4) = (A4-1)*[A4^3-5*A4^2+18*A4-8]/8
(A5) = A4+1	Etc.	Etc.

The results are as amazing! Here is a copy of what my spreadsheet produced.

Cards	Arrangements	Regions
1	1	1
2	1	2
3	2	7
4	14	18
5	252	41
6	10332	85
7	878220	162
8	142271640	287
9	40831960680	478
10	1.95177E+13	756
11	1.47554E+16	1145

12	1.68949E+19	1672
13	2.82483E+22	2367
14	6.68636E+25	3263
15	2.18176E+29	4396
16	9.59102E+32	5805
17	5.56759E+36	7532
18	4.19351E+40	9622
19	4.03499E+44	12123
20	4.89162E+48	15086
21	7.3795E+52	18565
22	1.37E+57	22617
23	3.09854E+61	27302
24	8.45963E+65	32683
25	2.76486E+70	38826
26	1.07348E+75	45800
27	4.91656E+79	53677
28	2.63906E+84	62532
29	1.65026E+89	72443
30	1.1955E+94	83491
31	9.98132E+98	95760
32	9.5581E+103	109337
33	1.0451E+109	124312
34	1.2991E+114	140778
35	1.8289E+119	158831
36	2.9048E+124	178570
37	5.1872E+129	200097
38	1.0379E+135	223517
39	2.32E+140	248938
40	5.7753E+145	276471
41	1.5967E+151	306230
42	4.8896E+156	338332
43	1.6543E+162	372897
44	6.1688E+167	410048
45	2.5295E+173	449911
46	1.1381E+179	492615
47	5.6062E+184	538292
48	3.0178E+190	587077
49	1.7717E+196	639108
50	1.1323E+202	694526
51	7.8641E+207	753475
52	5.9254E+213	816102

So, when you toss just five cards on the table or ground there are 252 possible arrangements.

With just nine cards, this number grows to 40,831,960,680 (over 40 billion) distinctly different arrangements!

When you play 52 Pickup, there are an astonishing 5.9254 × 10^{213} possible arrangements!

To give you some idea of the size of this number, imagine not just every galaxy in the universe (there are hundreds of billions of them) or even every star (there are hundreds of billions in each galaxy), or even all the atoms or ions that make up all the stars … try, as hard as you can, to imagine the sheer number of subatomic particles that make up all the atoms and ions, that make everything that we see in the entire visible universe. Throw in all the photons of light and neutrinos and the like for good measure.

Now, imagine that you could squash our entire universe down so that it fitted inside a subatomic particle and do THAT for each and every subatomic particle in our entire visible universe … each and every one of them … a complete universe inside every subatomic particle in another complete universe! That will give you about 10^{180} subatomic particles.

Now, shrink that new universe of universes to the size of a pin head (not one of those nice pearl ones … I mean the common industrial ones with the tiny heads) and reproduce it over and over and over again until you can completely replace the whole of planet earth right to the centre of the core … all of it. You are still not quite finished! You now need about 20,000 duplicate planet earths of that size!

Think of it! Every possible known subatomic particle in the vastness of our universe reproduced again inside every other subatomic particle in a strange new universe composed of universes, all reproduced in the each of the pinheads that compose roughly 20,000 entire planet earths! That is the number of possible arrangements when you toss 52 cards in the air!

What is more … if you allow them to line up or land exactly in top of each other, or count face-up and face-down as two different states, that number INCREASES!

This is extraordinary. What an amazing insight we have into such a simple act … the tossing of a deck of cards. I hope you are impressed with the vast number of possibilities that we encounter in everyday life.

Let me encourage you to think of how the coins may be arranged in your pocket, or notes in a wallet, or beads on a necklace.

This form of counting (combinatorics) is used in many endeavours in the world … in government, finance, insurance, and industry … but it is "behind the scenes."

With best wishes to you,

Graeme Henderson
01 May 2015

AN IMPORTANT NOTE!

I wish to thank Chris Aguilar for his kind permission to use his vectorised images of playing cards in my illustrations. His images are licenced under LGPL 3 - www.gnu.org/copyleft/lesser.html, © 2011 — Chris Aguilar. He has done a brilliant job and, should you need such images, or simply want to learn something about the process that he used, please visit https://code.google.com/p/vectorized-playing-cards/ (Vectorized Playing Cards 1.3).

